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# Embeddings of Beppo–Levi spaces in Hölder–Zygmund spaces, and a new method for radial basis function interpolation error estimates

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**Abstract**

The Beppo–Levi native spaces which arise when using polyharmonic splines to interpolate in many space dimensions are embedded in Hölder–Zygmund spaces. Convergence rates for radial basis function interpolation are inferred in some special cases.


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## 1. Introduction

A radial basis function is a function of the form

$$s = p + \sum_{x \in X} \gamma_x \Phi(\cdot - x), \tag{1}$$

where $p$ is a low degree polynomial, $\Phi$ is a fixed radially symmetric function, $X$ is a finite set of points and $\gamma_x$ is a coefficient associated with the point $x \in X$. The error analysis for radial basis function interpolation using polyharmonic splines takes place in Beppo–Levi spaces (see [3] and below for a definition). Our aim is to infer convergence rates for this interpolation from

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the knowledge that the Beppo–Levi seminorm of the interpolation error is bounded. We do this by seeing that the Beppo–Levi space can be embedded in some homogeneous Hölder–Zygmund space which implies that the first or second differences of certain derivatives of the error decay as $h$ to a power. Error estimates can then be deduced from the knowledge that the error is zero at the interpolation points.

We begin by introducing some standard concepts and notation. For a multi-index $\alpha \in \mathbb{N}^d$, define $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$ and $D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}$, where $\partial_s = \frac{\partial}{\partial x_s}$. Let $\mathcal{S}$ denote the space of rapidly decaying, infinitely differentiable functions on $\mathbb{R}^d$ and $\mathcal{S}'$ denote its dual space, the space of tempered distributions. We denote the action of a distribution $f$ on a test function $\rho$ by $\langle f, \rho \rangle$. For a function $\rho \in \mathcal{S}$ we define the Fourier transform

$$\rho^\wedge(x) = \int_{\mathbb{R}^d} \rho(y) e^{-ixy}\, dy.$$

Then, the Fourier transform of $f \in \mathcal{S}'$ is defined by

$$\langle f^\wedge, \rho \rangle = \langle f, \rho^\wedge \rangle \quad \text{for all } \rho \in \mathcal{S}.$$

The polyharmonic spline basic functions are given by

$$\Phi_{d,k}(x) = \begin{cases} |x|^{2k-d}, & d \text{ odd}, \\ |x|^{2k-d} \log |x|, & d \text{ even}, \end{cases} \tag{2}$$

where $2k > d$ and $|\cdot|$ denotes the Euclidean norm. The corresponding polyharmonic splines have the form

$$s = p_{k-1} + \sum_{x \in X} \gamma_x \Phi_{d,k}(\cdot - x), \tag{3}$$

where $p_{k-1} \in \pi_{k-1}^d$ the space of polynomials of degree $k-1$ in $d$ variables. The coefficients $\{\gamma_x\}$ will be described (in a conventional but regrettable notation) as orthogonal to $\pi_{k-1}^d$ in the sense that

$$L(q) := \sum_{x \in X} \gamma_x q(x) = 0 \quad \text{for all } q \in \pi_{k-1}^d. \tag{4}$$

For any open set $\Omega$, $\mathcal{D}(\Omega)$ denotes the space of all $C^\infty$ functions $\phi$ with compact support $K \subset \Omega$. Further, for any function $g \in L^1_{\text{loc}}(\Omega)$, the locally integrable functions on $\Omega$, let $\Lambda_{\Omega,g}$ be the distribution in $\mathcal{D}'(\Omega)$ defined by

$$\langle \Lambda_{\Omega,g}, \phi \rangle = \int_\Omega g(\xi)\phi(\xi)\, d\xi \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

$\Lambda_g$ is shorthand for $\Lambda_{\mathbb{R}^d,g}$. Often we will write $g$ instead of $\Lambda_g$.

The error analysis for interpolation using polyharmonic splines is made relatively straightforward by the fact that (distributionally)

$$\Delta_d^k \Phi_{d,k} = C_{d,k}\delta_0$$

for some constant $C_{d,k}$, where $\delta_0$ is the $d$-dimensional Dirac measure at the origin. A good reference for these matters is Gelfand–Shilov [6].

The analysis of radial basis interpolation in $d$-dimensional space using polyharmonic splines takes place in Beppo–Levi spaces, also known as homogeneous Sobolev spaces. For $k \in \mathbb{Z}_+$ the Beppo–Levi spaces $\mathrm{BL}_k(\mathbb{R}^d)$ is defined to be the space of all tempered distributions $f$ on $\mathbb{R}^d$ such that $D^\alpha f \in L^2(\mathbb{R}^d)$ for all $|\alpha| = k$; see Deny and Lions [3, p. 366]. We will shorten this notation to $\mathrm{BL}_k$ when there is no danger of confusion. A seminorm on this space is

$$|f|_{\mathrm{BL}_k} = \left\{ \sum_{|\alpha|=k} c_\alpha \|D^\alpha f\|_2^2 \right\}^{1/2}, \tag{5}$$

where $c_\alpha = \frac{k!}{\alpha_1!\ldots\alpha_d!}$ and by the Multinomial Formula

$$\sum_{|\alpha|=k} c_\alpha x^{2\alpha} = |x|^{2k}.$$

The kernel of this seminorm is just $\pi_{k-1}^d$. By an embedding theorem for Beppo–Levi spaces, when $2k > d$ elements of $\mathrm{BL}_k(\mathbb{R}^d)$ are continuous functions (see e.g. Duchon [4]).

Define the subspace $\mathcal{S}_{k-1}$ of $\mathcal{S}$ by

$$\mathcal{S}_{k-1} = \left\{ \psi \in \mathcal{S} : \int_{\mathbb{R}^d} x^\beta \psi(x)\, dx = 0 \quad \text{for all } |\beta| \leqslant k-1 \right\}$$

and let

$$\widehat{\mathcal{S}_{k-1}} = \left\{ \phi : \phi = \widehat{\psi} \text{ for some } \psi \in \mathcal{S}_{k-1} \right\}.$$

Then both $\mathcal{S}_{k-1}$ and $\widehat{\mathcal{S}_{k-1}}$ are subspaces of $\mathcal{S}$. We equip them with the topology of $\mathcal{S}$.

Note that if $\phi \in \widehat{\mathcal{S}_{k-1}}$, then $(D^\beta \phi)(0) = 0$ for all $|\beta| \leqslant k-1$, so that by Taylor polynomial expansion

$$|\phi(\xi)|/|\xi|^k \leqslant C \sum_{|\alpha|=k} \|D^\alpha \phi\|_\infty \quad \text{for all } \xi \neq 0.$$

Since also $\phi \in \mathcal{S}$ it follows that $\phi/|\cdot|^k \in L^2(\mathbb{R}^d)$. Furthermore,

$$\phi_j \to 0 \text{ in } \widehat{\mathcal{S}_{k-1}} \quad \text{implies} \quad \phi_j/|\cdot|^k \to 0 \text{ in } L^2(\mathbb{R}^d). \tag{6}$$

We will need the following lemma concerning functions in a Beppo–Levi space.

**Lemma 1.** *Let $k \in \mathbb{N}$ and $f \in \mathrm{BL}_k$. Then there exists a function $g \in L^1_{\mathrm{loc}}\left(\mathbb{R}^d \backslash \{0\}\right)$ such that $\||\cdot|^k g\|_2 = |f|_{\mathrm{BL}_k}$, and*

$$\langle \widehat{f}, \phi \rangle = \int_{\mathbb{R}^d} g(\xi)\phi(\xi)\, d\xi \quad \textit{for all } \phi \in \widehat{\mathcal{S}_{k-1}}.$$

**Proof.** Let $\alpha$ be a multiindex with $|\alpha| = k$. Then

$$(D^\alpha f)^\wedge = (\mathrm{i}\,\xi)^\alpha \widehat{f} = \Lambda_{g_\alpha} \tag{7}$$

for some $g_\alpha \in L^2(\mathbb{R}^d) \subset L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Given $0 \neq x \in \mathbb{R}^d$, choose $j$ so $|x_j| = \|x\|_\infty$. Choose $\beta$ as the multiindex with $j$th component $k$ and other components zero. Then $(\mathrm{i}\,\xi)^\beta$ is bounded away from zero on the open ball $U_x$ about $x$ of radius $\|x\|_\infty/2$. Let $\phi \in \mathcal{D}(U_x)$ and note $1/(\mathrm{i}\,\xi)^\beta \in C^\infty(U_x) \cap L^\infty(U_x)$. Hence $\frac{1}{(\mathrm{i}\,\xi)^\beta}\phi \in \mathcal{D}(U_x)$ and

$$\begin{aligned}\langle \widehat{f}, \phi\rangle &= \left\langle (\mathrm{i}\,\xi)^\beta \widehat{f}, \frac{1}{(\mathrm{i}\,\xi)^\beta}\phi \right\rangle \\ &= \int_{U_x} \left( g_\beta(\xi) \frac{1}{(\mathrm{i}\,\xi)^\beta}\right) \phi(\xi)\, d\xi \\ &= \int_{U_x} h_{U_x}(\xi)\phi(\xi)\, d\xi,\end{aligned}$$

where $h_{U_x}(\xi) := \frac{1}{(\mathrm{i}\,\xi)^\beta} g_\beta(\xi)$ for all $\xi \in U_x$, is a function in $L^1(U_x)$. Now suppose $x, y$ in $\mathbb{R}^d$ are such that $V = U_x \cap U_y$ is nonempty. Then

$$\int_{\mathbb{R}^d} h_{U_x}(\xi)\phi(\xi)\, d\xi = \langle \widehat{f}, \phi\rangle = \int_{\mathbb{R}^d} h_{U_y}(\xi)\phi(\xi)\, d\xi$$

for all $\phi \in \mathcal{D}(V)$, which implies $h_{U_x} = h_{U_y}$ a.e. on $V$. Define $g = h_{U_x}$ on $U_x$. Then $g$ is well defined and is locally integrable on $\mathbb{R}^d\backslash\{0\}$. By a partition of unity argument we see that

$$\langle \widehat{f}, \phi\rangle = \int_{\mathbb{R}^d} g(\xi)\phi(\xi)\, d\xi \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}^d\backslash\{0\}). \tag{8}$$

By (7) this implies that for every $|\alpha| = k$,

$$(\mathrm{i}\,\xi)^\alpha g(\xi) = g_\alpha(\xi) \quad \text{a.e. on } \mathbb{R}^d. \tag{9}$$

Since $\sum_{|\alpha|=k} c_\alpha \xi^{2\alpha} = |\xi|^{2k}$ for all $\xi \in \mathbb{R}^d$,

$$\left\| |\xi|^k g \right\|_2^2 = \left\| \sum_{|\alpha|=k} c_\alpha \xi^{2\alpha} |g(\xi)|^2 \right\|_1 = \sum_{|\alpha|=k} c_\alpha \|g_\alpha\|_2^2,$$

where in the last step we have used (9). It follows that

$$\left\| |\cdot|^k g \right\|_2^2 = \sum_{|\alpha|=k} c_\alpha \|g_\alpha\|_2^2 = |f|^2_{\mathrm{BL}_k}. \tag{10}$$

Now define a functional $\widehat{f_1}$ on $\widehat{\mathcal{S}_{k-1}}$ by

$$\langle \widehat{f_1}, \phi\rangle = \int_{\mathbb{R}^d} g(\xi)\phi(\xi)\, d\xi = \int_{\mathbb{R}^d} \left\{ |\xi|^k g(\xi) \right\} \left\{ \frac{\phi(\xi)}{|\xi|^k} \right\} d\xi. \tag{11}$$

Then by (6), $\widehat{f_1}$ is continuous on $\widehat{\mathcal{S}_{k-1}}$. By the Hahn–Banach theorem $\widehat{f_1}$ can be extended to a continuous linear functional on $\mathcal{S}$.

Then for $\phi \in \mathcal{S}$, and multiindex $\alpha$ with $|\alpha| = k$, $(\mathrm{i}\,\xi)^\alpha \phi(\xi) \in \widehat{\mathcal{S}_{k-1}}$ and

$$\begin{aligned}\langle (D^\alpha f_1)^\wedge, \phi\rangle &= \langle \widehat{f_1}, (\mathrm{i}\,\xi)^\alpha \phi\rangle \\ &= \int_{\mathbb{R}^d} g(\xi)(\mathrm{i}\,\xi)^\alpha \phi(\xi)\, d\xi = \int_{\mathbb{R}^d} g_\alpha(\xi)\phi(\xi)\, d\xi.\end{aligned}$$

It follows that $(D^{\alpha} f_1)^{\wedge} = g_{\alpha}$ in $\mathcal{S}'$, and

$$|f_1|^2_{\mathrm{BL}_k} = \sum_{|\alpha|=k} c_{\alpha} \|D^{\alpha} f_1\|_2^2 = \sum_{|\alpha|=k} c_{\alpha} \|g_{\alpha}\|_2^2 = |f|^2_{\mathrm{BL}_k}.$$

Since $\mathcal{D}(\mathbb{R}^d \backslash \{0\}) \subset \widehat{\mathcal{S}_{k-1}}$, (8) and (11) imply $\langle \widehat{f_1}, \phi \rangle = \langle \widehat{f}, \phi \rangle$ for all $\phi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$. Hence $\widehat{f - f_1}$ is supported at the origin, so that $f - f_1$ is a polynomial say $p$. Since $f - f_1 \in \mathrm{BL}_k$ the polynomial $p$ is in $\pi^d_{k-1}$. Since $p \in \pi^d_{k-1}$

$$\langle \widehat{p}, \phi \rangle = 0 \quad \text{for all } \phi \in \widehat{\mathcal{S}_{k-1}},$$

so that

$$\langle \widehat{f}, \phi \rangle = \langle \widehat{f_1}, \phi \rangle = \int_{\mathbb{R}^d} g(\xi) \phi(\xi)\, d\xi \quad \text{for all } \phi \in \widehat{\mathcal{S}_{k-1}} \tag{8$'$}$$

and the lemma follows. □

## 2. A scale of Hölder–Zygmund spaces

Let $0 < \sigma \leqslant 1$. We define the (homogeneous) Hölder–Zygmund space $\dot{\mathcal{C}}^{\sigma}$ to be the space of all continuous functions $f : \mathbb{R}^d \to \mathbb{C}$ such that the seminorm

$$\|f\|_{\dot{\mathcal{C}}^{\sigma}} := \begin{cases} \sup\limits_{0 \neq h \in \mathbb{R}^d} \dfrac{\|\Delta_h f\|_{\infty}}{|h|^{\sigma}}, & 0 < \sigma < 1, \\[2ex] \sup\limits_{0 \neq h \in \mathbb{R}^d} \dfrac{\left\|\Delta_h^2 f\right\|_{\infty}}{|h|}, & \sigma = 1, \end{cases} \tag{12}$$

is finite. Here $\Delta_h f(x)$ is the forward difference $\Delta_h^1 f(x) = f(x+h) - f(x)$, and for $n = 2, 3, \ldots, \Delta_h^n f = \Delta_h \left( \Delta_h^{n-1} f \right)$.

Now consider $\sigma > 1$. Let $j$ be the greatest nonnegative integer such that $j < \sigma$ which implies $0 < \sigma - j \leqslant 1$. Define $\dot{\mathcal{C}}^{\sigma}$ to be the space of all $C^j$ functions such that

$$\|f\|_{\dot{\mathcal{C}}^{\sigma}} := \sum_{|\alpha|=j} \|D^{\alpha} f\|_{\dot{\mathcal{C}}^{\sigma-j}} < \infty. \tag{13}$$

Let $[\cdot]$ denote the integer part function. Then we combine (12) and (13) to rewrite the definition of the seminorm in an expression valid for all $0 < \sigma$

$$\|f\|_{\dot{\mathcal{C}}^{\sigma}} = \begin{cases} \sum\limits_{|\alpha|=[\sigma]} \sup\limits_{0 \neq h \in \mathbb{R}^d} \dfrac{\left\|\Delta_h \left(D^{\alpha} f\right)\right\|_{\infty}}{|h|^{\sigma-[\sigma]}}, & \sigma \text{ non integer}, \\[2ex] \sum\limits_{|\alpha|=\sigma-1} \sup\limits_{0 \neq h \in \mathbb{R}^d} \dfrac{\left\|\Delta_h^2 \left(D^{\alpha} f\right)\right\|_{\infty}}{|h|}, & \sigma \text{ an integer}. \end{cases} \tag{14}$$

The kernel of the seminorm is $\pi^d_{[\sigma]}$.

It is interesting to recall one of Zygmund's [12] reasons for introducing the Zygmund space of $2\pi$ periodic functions $g$ with second modulus

$$\omega_2(g, h) := \sup_{x\in\mathbb{R},\ 0<k\leqslant h} |g(x) - 2g(x+k) + g(x+2k)|,$$

of order $\mathcal{O}(h)$ as $h \to 0$. This was that the Lipschitz classes associated with the ordinary modulus $\omega(g, h)$ do not characterize those functions $g$ such that the error in best approximation by trigonometric polynomials of degree $n$, $E_n^\star(g)$ is $\mathcal{O}(n^{-1})$ as $n \to \infty$. Indeed using the ordinary modulus one has

$$\omega(g, h) = \mathcal{O}(h) \implies E_n^\star(g) = \mathcal{O}(n^{-1}) \implies \omega(g, h) = \mathcal{O}(h|\log h|),$$

whereas using the second modulus, that is a Zygmund space, one has

$$\omega_2(f, h) = \mathcal{O}(h) \quad \iff \quad E_n^\star(g) = \mathcal{O}(n^{-1}).$$

In view of this history we call the spaces $\dot{\mathcal{C}}^\sigma$ Hölder–Zygmund spaces. These spaces have also played an important role in other branches of analysis such as harmonic analysis and partial differential equations.

There is a family of equivalent seminorms for $\dot{\mathcal{C}}^\sigma$. Let $0 < \sigma < n + j$ where $n$ and $j$ are nonnegative integers such that $0 \leqslant j < \sigma$. Then

$$\|f\|_{\dot{\mathcal{C}}^\sigma} \approx \sum_{|\alpha|=j} \sup_{0\neq h\in\mathbb{R}^d} \frac{\left\|\Delta_h^n\left(D^\alpha f\right)\right\|_\infty}{|h|^{\sigma-j}} \tag{15}$$

for all $f \in \dot{\mathcal{C}}^\sigma$. Note that the right-hand side of (15) contains (14) as a special case with $j$ the greatest integer less than $\sigma$ and $n$ chosen as 2 or 1 according as $\sigma$ is, or is not, an integer. The right-hand side of (15) has kernel $\pi_{n+j-1}^d$ and without the restriction $f \in \dot{\mathcal{C}}^\sigma$ the equivalence should be interpreted modulo $\pi_{n+j-1}^d$.

## 3. The embedding theorem

The embedding theorem we prove in this section is a folklore result in harmonic analysis. Usually such theorems would be proven by means of a Littlewood–Paley decomposition. (We refer to [1,2,8,10] for the Littlewood–Paley theory.) An inconvenient aspect of the Littlewood–Paley norms is that they annihilate polynomials of all orders, while we want to control the degrees of the polynomials appearing in the embedding theorem, and this would usually require additional arguments. However in the 2-norm setting that we are interested in it is possible to give a simple direct proof which we present in the next theorem.

**Theorem 2.** *Suppose $k > d/2$ and define $\sigma = k - \frac{d}{2}$. Suppose $n \in \mathbb{N}$, $j \in \mathbb{N}_0$ are such that $0 \leqslant j < \sigma$ and $0 < \max\{\sigma, k-1\} < n + j$. Then there exists a constant $E$, depending on $k$, $d$, $n$ and $j$, such that*

$$\sup_{0\neq h\in\mathbb{R}^d} \sum_{|\alpha|=j} |h|^{j-\sigma} \|\Delta_h^n\left(D^\alpha f\right)\|_\infty \leqslant E|f|_{\mathrm{BL}_k} \quad \textit{for all } f \in \mathrm{BL}_k. \tag{16}$$

*Furthermore*, $\mathrm{BL}_k$ *is continuously embedded in* $\dot{\mathcal{C}}^\sigma$ *modulo* $\pi_{k-1}^d$. *Thus there exists a constant G depending only on k and d such that for each* $f \in \mathrm{BL}_k$ *there is a corresponding polynomial q in* $\pi_{k-1}^d$ *so that*

$$\|f-q\|_{\dot{\mathcal{C}}^{k-\frac{d}{2}}} \leqslant G|f|_{\mathrm{BL}_k}.$$

**Proof.** Let $f \in \mathrm{BL}_k$ and $\alpha$ be a multiindex with $|\alpha| = j$. Let $g$ be the $L^1_{\mathrm{loc}}\left(\mathbb{R}^d\backslash\{0\}\right)$ function whose existence is guaranteed by Lemma 1. We will show shortly that the function

$$m(\xi) = g(\xi)(e^{\mathrm{i}\,\xi h}-1)^n(\mathrm{i}\,\xi)^\alpha \tag{17}$$

is in $L^1(\mathbb{R}^d)$ with

$$\|m\|_1 \leqslant C|h|^{\sigma-j}|f|_{\mathrm{BL}_k}. \tag{18}$$

Assume this in the meantime.

Begin with the equation

$$\left\langle\left[\Delta_h^n\left(D^\alpha f\right)\right]^\wedge, \phi\right\rangle = \langle\widehat{f}, (e^{\mathrm{i}\,\xi h}-1)^n(\mathrm{i}\,\xi)^\alpha\phi(\xi)\rangle \quad \text{for all } \phi \in \mathcal{S}. \tag{19}$$

Note that for $|\xi||h| \leqslant 1$, $\left|e^{\mathrm{i}\,\xi h}-1\right| \leqslant |\xi||h|$. Since, by hypothesis, $n + j \geqslant k$ then for $\phi \in \mathcal{S}$, $(e^{\mathrm{i}\,\xi h}-1)^n(\mathrm{i}\,\xi)^\alpha\phi(\xi)$ is in $\widehat{\mathcal{S}_{k-1}}$ and, by Lemma 1, Eq. (19) can be realized as

$$\left\langle\left[\Delta_h^n\left(D^\alpha f\right)\right]^\wedge, \phi\right\rangle = \int_{\mathbb{R}^d} g(\xi)(e^{\mathrm{i}\,\xi h}-1)^n(\mathrm{i}\,\xi)^\alpha\phi(\xi)\,d\xi = \int_{\mathbb{R}^d} m(\xi)\phi(\xi)\,d\xi. \tag{20}$$

Therefore by Fourier inversion, and the assumed estimate for $m$, (18),

$$\left\|\Delta_h^n\left(D^\alpha f\right)\right\|_\infty \leqslant (2\pi)^{-d}\|m\|_1 \leqslant C|h|^{\sigma-j}|f|_{\mathrm{BL}_k},$$

and estimate (16) follows.

We now proceed to show that the function $m$ of Eq. (17) satisfies estimate (18).

$$\begin{aligned}
\|m\|_1 &= \int_{\mathbb{R}^d}\left|g(\xi)\left(e^{\mathrm{i}\,\xi h}-1\right)^n(\mathrm{i}\,\xi)^\alpha\right|d\xi \\
&= \int_{\mathbb{R}^d}\left|\frac{\left(e^{\mathrm{i}\,\xi h}-1\right)^n}{|\xi|^k}(\mathrm{i}\,\xi)^\alpha g(\xi)|\xi|^k\right|d\xi \\
&\leqslant \left\|\frac{\left(e^{\mathrm{i}\,\xi h}-1\right)^n(\mathrm{i}\,\xi)^\alpha}{|\cdot|^k}\right\|_2 \left\|g|\cdot|^k\right\|_2 \\
&= \left\|\frac{\left(e^{\mathrm{i}\,\xi h}-1\right)^n(\mathrm{i}\,\xi)^\alpha}{|\cdot|^k}\right\|_2 |f|_{\mathrm{BL}_k},
\end{aligned} \tag{21}$$

where in the last step we have used Lemma 1.

The only thing that remains is to estimate the first term on the right-hand side of Eq. (21).

Consider two separate cases $|\xi||h| \leqslant 1$ and $|\xi||h| > 1$. In the first case the inequality

$$\left|e^{\mathrm{i}\,\xi h} - 1\right| \leqslant |\xi||h|$$

holds. Applying this we get

$$\left|\frac{\left(e^{\mathrm{i}\,\xi h} - 1\right)^n (\mathrm{i}\,\xi)^\alpha}{|\xi|^k}\right| \leqslant \frac{|\xi|^n |h|^n |\xi|^j}{|\xi|^k} = |h|^n |\xi|^{n+j-k}.$$

Hence, with $\omega_d$ the surface area of the unit sphere in $\mathbb{R}^d$,

$$\begin{aligned}
\left(\int_{|\xi| \leqslant 1/|h|} \left|\frac{\left(e^{\mathrm{i}\,\xi h} - 1\right)^n (\mathrm{i}\,\xi)^\alpha}{|\xi|^k}\right|^2 d\xi\right)^{\frac{1}{2}} &\leqslant \sqrt{\omega_d}\, |h|^n \left(\int_0^{1/|h|} r^{2(n+j-k)+d-1}\, dr\right)^{\frac{1}{2}} \\
&= \frac{\sqrt{\omega_d}\, |h|^n}{\sqrt{2n+2j-2\sigma}} \left(\frac{1}{|h|}\right)^{n+j-\sigma} \\
&= \frac{\sqrt{\omega_d}}{\sqrt{2n+2j-2\sigma}} |h|^{\sigma-j}.
\end{aligned}$$

When $|\xi||h| > 1$ use the estimate $\left|e^{\mathrm{i}\,\xi h} - 1\right| \leqslant 2$. Then

$$\left|\frac{\left(e^{\mathrm{i}\,\xi h} - 1\right)^n (\mathrm{i}\,\xi)^\alpha}{|\xi|^k}\right| \leqslant 2^n |\xi|^{j-k}$$

so that

$$\begin{aligned}
\left(\int_{|\xi| > \frac{1}{|h|}} \left|\frac{\left(e^{\mathrm{i}\,\xi h} - 1\right)^n (\mathrm{i}\,\xi)^\alpha}{|\xi|^k}\right|^2 d\xi\right)^{\frac{1}{2}} &\leqslant 2^n \sqrt{\omega_d} \left(\int_{\frac{1}{|h|}}^{\infty} r^{2(j-k)+d-1}\, dr\right)^{\frac{1}{2}} \\
&= \frac{2^n \sqrt{\omega_d}}{\sqrt{-d-2(j-k)}} \left(\frac{1}{|h|}\right)^{j-k+\frac{d}{2}} \\
&= \frac{2^n \sqrt{\omega_d}}{\sqrt{2\sigma - 2j}} |h|^{\sigma-j}.
\end{aligned}$$

This completes the proof of the first statement in the theorem.

We now turn to the second statement of theorem which concerns the continuous embedding of $\mathrm{BL}_k$ in $\dot{C}^\sigma$ modulo $\pi_{k-1}^d$. For this choose $0 \leqslant j < \sigma$ and $n$ such that $n + j = k$. Then from the previous result

$$\sum_{|\alpha|=j} |h|^{j-\sigma} \|\Delta_h^n \left(D^\alpha f\right)\|_\infty \leqslant E |f|_{\mathrm{BL}_k}.$$

The quantity on the left above corresponds to one of the equivalent seminorms discussed in the section on Hölder–Zygmund spaces, the equivalence to be interpreted modulo $\pi_{n+j-1}^d = \pi_{k-1}^d$.

That is there exist a polynomial $q \in \pi_{k-1}^d$ such that

$$\|f-q\|_{\dot{C}^{k-\frac{d}{2}}} \leqslant G|f|_{\mathrm{BL}_k},$$

completing the proof of the theorem. $\square$

## 4. Interpolation and error estimates

Consider the following interpolation problem. Let $X \subset \mathbb{R}^d$ be a finite set of distinct points unisolvent for $\pi_{k-1}^d$. Let $s$ be the function which interpolates to $f \in \mathrm{BL}_k$ on $X$ and has the form

$$s = p_{k-1} + \sum_{x\in X} \gamma_x \Phi_{d,k}(\cdot - x), \tag{22}$$

where $p_{k-1} \in \pi_{k-1}^d$. In order that $s$ be in $\mathrm{BL}_k$ we require that the coefficients $\{\gamma_x\}$ be orthogonal to $\pi_{k-1}^d$ in the sense defined in (4). The existence and uniqueness of such a polyharmonic spline interpolant is well known.

Then, using standard arguments (see e.g. [5, Lemma 3.2]) one can easily show that, for any $g \in \mathrm{BL}_k$, with $g(x) = f(x)$, for all $x \in X$,

$$|g|^2_{\mathrm{BL}_k} = |s|^2_{\mathrm{BL}_k} + |g-s|^2_{\mathrm{BL}_k},$$

so that $s$ minimizes the seminorm (energy functional) over all interpolants from $\mathrm{BL}_k$. In particular $|f-s|_{\mathrm{BL}_k} \leqslant |f|_{\mathrm{BL}_k}$.

Now, applying the embedding of Theorem 2, and in particular (16), we find

**Corollary 3.** *Suppose $k > d/2$ and define $\sigma = k - \frac{d}{2}$. Suppose $n \in \mathbb{N}$, $j \in \mathbb{N}_0$ are such that $0 \leqslant j < \sigma$ and $0 < \max\{\sigma, k-1\} < n+j$. Let $X$ be a finite set of distinct points in $\mathbb{R}^d$ unisolvent for $\pi_{k-1}^d$. Then there exists a constant $C$, depending only on $k$, $d$, $n$ and $j$, such that if $f \in \mathrm{BL}_k$ and $s$ is the polyharmonic spline interpolant to $f$ on $X$ of form* (22) *then*

$$\sup_{0\neq h\in\mathbb{R}^d} \sum_{|\alpha|=j} |h|^{j-\sigma} \|\Delta_h^n \left(D^\alpha(f-s)\right)\|_\infty \leqslant C|f|_{\mathrm{BL}_k}. \tag{23}$$

Corollary 3 can be used to produce error estimates for polyharmonic spline interpolation.

## 5. Examples

We present three examples here. The first gives convergence rates in one dimension for natural splines of odd degree. The second is for thin-plate spline interpolation in two dimensions, and the third for interpolation using $|x|$ in three dimensions.

### *5.1. Univariate splines*

In this case, for $k \in \mathbb{N}$, we consider the basic function $\Phi_{1,k}(x) = |x|^{2k-1}$. Let $X \subset [a,b] \subset \mathbb{R}$, with $a, b \in X$. The natural degree $2k-1$ spline interpolant has the form

$$s = p_{k-1} + \sum_{x\in X} \gamma_x |\cdot - x|^{2k-1},$$

where $p_{k-1} \in \pi^1_{k-1}$. The set $X$ is required to be unisolvent for $\pi^d_{k-1}$ which in this one-dimensional case reduces to $X$ having cardinality, $\#X$, at least $k$. The coefficients $\{\gamma_x : x \in X\}$ are required to be orthogonal to $\pi^1_{k-1}$. The native Beppo–Levi space for $\Phi_{1,k}$ is $\mathrm{BL}_k(\mathbb{R})$ which by Theorem 2 is embedded in $\dot{\mathcal{C}}_{k-1/2}(\mathbb{R})$ modulo $\pi^1_{k-1}$. Further, Corollary 3 implies the existence of a constant $C$ such that

$$\sup_{0\neq h\in\mathbb{R}} |h|^{-1/2}\|\Delta_h D^{k-1}[f-s]\|_\infty \leqslant C|f|_{\mathrm{BL}_k(\mathbb{R})}. \tag{24}$$

Recall that a set $X$ is said to have separation distance $\rho$ for a set $Y$ if

$$\sup_{y\in Y}\inf_{x\in X}|y-x| = \rho.$$

Let $\rho$ be the separation distance for the set $X$ in $[a, b]$. Label the points in $X$ in increasing order as

$$a = x_1 < x_2 < \cdots < x_n = b.$$

Applying Rolle's Theorem repeatedly, as in the classical proof of the formula for the error in polynomial interpolation, each interval $(x_i, x_{i+m})$ contains at least one zero of $D^{(m)}[f-s]$, for $1\leqslant m < k$ and $1\leqslant i\leqslant n-m$. Denote the set of zeros of $D^m[f-s]$ in $[a,b]$ by $X_m$. The Rolle's Theorem argument implies that for each $1\leqslant m < k$ the set $X_m$ has cardinality at least $\#X - m$ and considered as a subset of $[a, b]$ has a separation distance not exceeding $2m\rho$.

Now, let $z \in [a, b]$ be such that $M = \max_{y\in[a,b]} |D^{k-1}[f-s](y)| = |D^{k-1}[f-s](z)|$. Let $\xi$ be the nearest point to $z$ in $X_{k-1}$. Then necessarily $|z-\xi| \leqslant 2(k-1)\rho =: h$. Hence, using (24),

$$M = |D^{k-1}[f-s](z) - D^{k-1}[f-s](\xi)| \leqslant Ch^{1/2}|f|_{\mathrm{BL}_k(\mathbb{R})} \leqslant C'\rho^{1/2}|f|_{\mathrm{BL}_k(\mathbb{R})}.$$

Since $D^{k-2}[f-s]$ is differentiable on $[a, b]$, the Mean Value Theorem implies that for any $y$ and $\xi$ in $[a, b]$

$$D^{k-2}[f-s](y) - D^{k-2}[f-s](\xi) = (y-\xi)D^{k-1}[f-s](c),$$

where $c$ is an unknown point between $y$ and $\xi$. Now, $D^{k-2}[f-s] = 0$ on the set of points $X_{k-2}$ which considered as a subset of $[a, b]$ has separation distance $2(k-2)\rho$. Hence choosing $\xi$ as the closest point to $y$ in $X_{k-2}$ it follows that

$$\sup_{y\in[a,b]} |D^{k-2}[f-s](y)| \leqslant C\rho^{3/2}|f|_{\mathrm{BL}_k(\mathbb{R})}.$$

We may proceed in the same way, using the Mean Value Theorem repeatedly, decreasing the order of the derivatives as we go, to finally conclude that

$$\|f-s\|_{\infty,[a,b]} \leqslant C\rho^{k-1/2}|f|_{\mathrm{BL}_k(\mathbb{R})}.$$

This agrees with the results given for example by Light and Wayne [7, Corollary 4.5].

We can also infer convergence rates for complete spline interpolation. Here, the spline interpolant is of the form

$$s = p_{k-1} + \sum_{x\in X}\gamma_x|\cdot - x|^{2k-1},$$

where $p_{k-1} \in \pi^1_{k-1}$. The coefficients are chosen so that $s(x) = f(x)$ for all $x \in X$ which contains $a$ and $b$, and also

$$D^m[f-s](a) = D^m[f-s](b) = 0, \quad 1 \leqslant m \leqslant k-1. \tag{25}$$

The complete spline end conditions (25) replace the orthogonality conditions (4) of the "natural" spline case. The complete spline also has a minimum energy characterization. It minimizes the energy seminorm $(\int_a^b [g^{(k)}(t)]^2\, dt)^{1/2}$ over all suitably smooth functions $g$ satisfying both the Lagrange interpolation conditions and the complete spline end conditions (25).

The native space in this case is the Sobolev space $W_k([a,b])$, whose elements are the restrictions of functions in $W_k(\mathbb{R})$ to $[a,b]$. These spaces are embedded in the inhomogeneous Hölder–Zygmund space of continuous functions whose $(k-1)$st derivative is in $\mathrm{Lip}_{1/2}[\mathrm{a, b}]$ (see e.g. [8]). Applying the same argument as for natural splines we infer a convergence rate of order $\rho^{k-\frac{1}{2}}$ where $2\rho$ is the mesh size.

### 5.2. *Thin-plate spline interpolation in two dimensions*

The minimal energy characterization of of thin-plate spline interpolants, due to Duchon [5], has influenced many to study and use radial basis functions. The simplest case corresponds to the displacement of a thin plate constrained to pass through certain points. Here the basic function is $\Phi_{2,2} = |\cdot|^2 \log|\cdot|$, mapping $\mathbb{R}^2 \to \mathbb{R}$ and the thin-plate spline interpolant has the form

$$s = p_1 + \sum_{x \in X} \gamma_x |\cdot - x|^2 \log|\cdot - x|,$$

where $p_1 \in \pi^2_1$ and the coefficients $\{\gamma_x : x \in X\}$ are orthogonal to linears. It minimizes the (linearized) bending energy over all sufficiently smooth interpolants, that is over all interpolants from $\mathrm{BL}_2(\mathbb{R}^2)$. We will show that there is an absolute constant $C$ such that if $s$ is the thin plate spline interpolant to $f \in \mathrm{BL}_2(\mathbb{R}^2)$ at nodes $X$ then on any closed triangle $T$ corresponding to three of the nodes

$$\|f - s\|_{\infty,T} \leqslant C\rho |f|_{\mathrm{BL}_2(\mathbb{R}^2)},$$

where $\rho$ is the length of the longest side of the triangle $T$. This agrees with the results given separately by Duchon [5], Light and Wayne [7], Powell [9], and Wu and Schaback [11].

The associated native Beppo–Levi space $\mathrm{BL}_2(\mathbb{R}^2)$ is embedded in $\dot{\mathcal{C}}_1(\mathbb{R}^2)$ modulo $\pi^2_1$. Further, Corollary 3 tells us that there exists a constant $C$ such that

$$\sup_{0 \neq h \in \mathbb{R}^2} |h|^{-1} \|\Delta_h^2[f-s]\|_\infty \leqslant C|f|_{\mathrm{BL}_2(\mathbb{R}^2)}. \tag{26}$$

Consider firstly the situation for a triangle $T$ with vertices interpolation nodes and an edge $[a,b]$ of $T$. Suppose that the maximum error on this edge, $M$, occurs at $z$, that is $M = \max_{y \in [a,b]} |[f - s](y)| = |[f-s](z)|$. Let $\xi$ be the nearest of the endpoints of the edge to $z$, and $h = z - \xi$. Then, $\Delta_h^2 g(\xi) = g(\xi) - 2g(z) + g(2z - \xi)$. Now using that $[f-s](\xi) = 0$ and (26),

$$\begin{aligned} |h|^{-1}|\Delta_h^2[f-s](\xi)| &= |h|^{-1}|2[f-s](z) - [f-s](2z-\xi)| \\ &\leqslant C|f|_{\mathrm{BL}_2(\mathbb{R}^2)}. \end{aligned}$$

Since $|[f-s](z)| = M$ and $|[f-s](2z-\xi)| \leqslant M$ it follows that

$$M \leqslant C|h||f|_{\mathrm{BL}_2(\mathbb{R}^2)} \leqslant C\frac{\rho}{2}|f|_{\mathrm{BL}_2(\mathbb{R}^2)}.$$

This establishes the bound on the error on the boundary of the triangle $T$.

Consider now the whole triangle $T$. Suppose that the maximum error on the triangle $M$ occurs at $z$. That is $M = \max_{y\in T} |[f-s](y)| = |[f-s](z)|$. If $z$ is on the boundary of $T$ the result has already been proven. Hence assume it does not. Let $\xi$ be the nearest point on the boundary of $T$ to $z$ and $h = z - \xi$. Then, again from Corollary 3,

$$\begin{aligned} |\Delta_h^2[f-s](\xi)| &= |[f-s](\xi) - 2[f-s](z) + [f-s](2z-\xi)| \\ &\leqslant C|h||f|_{\mathrm{BL}_2(\mathbb{R}^2)}. \end{aligned}$$

Thus, using the result already obtained for edges

$$M \leqslant |2[f-s](z) - [f-s](2z-\xi)| \leqslant C|h||f|_{\mathrm{BL}_2(\mathbb{R}^2)} + C\frac{\rho}{2}|f|_{\mathrm{BL}_2(\mathbb{R}^2)},$$

which gives the result for the whole of $T$.

### *5.3. Norm interpolation in three dimensions*

This example is similar to the last. The basic function is $\Phi_{2,3}(x) = |x|$, and the native Beppo–Levi space is $\mathrm{BL}_2(\mathbb{R}^3)$. The interpolant we seek is of the form

$$s = p_1 + \sum_{x\in X} \gamma_x |\cdot - x|,$$

where $p_1 \in \pi_1^3$. The coefficients $\{\gamma_x : x \in X\}$ are orthogonal to linears. The native Beppo–Levi space is embedded in $\dot{\mathcal{C}}^{1/2}(\mathbb{R}^3)$ modulo $\pi_1^3$. In particular Corollary 3 tells us that there exists a constant $C$ such that

$$\sup_{0\neq h\in\mathbb{R}^3} |h|^{-1/2} \|\Delta_h^2[f-s]\|_\infty \leqslant C|f|_{\mathrm{BL}_2(\mathbb{R}^3)}. \tag{27}$$

A simple argument then shows that there exists a constant $E$ such that if $z$ is any point in a closed tetrahedron with vertices interpolation nodes, and $h$ is the maximum length of an edge of the tetrahedron, then

$$|\,(f-s)\,(z)| \leqslant E h^{1/2} |f|_{\mathrm{BL}_2(\mathbb{R}^3)}.$$

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